Note: $\mathcal{R}[a,b]$:= Riemann integrable functions on $[a,b]$.

Q1. *(10 marks)* Give an example of a bounded function on $[a,b]$, $a<b$ which is not Riemann integrable.

(*Note: We can choose a highly discontinuous bounded function that will not be integrable.*)

Let $f:[a,b]\to\mathbb{R}$ be the function defined by $f(x)=x$ (or 1) for $x\in\mathbb{Q}\cap[a,b]$ and $f(x)=0$ for $x\in[a,b]-\mathbb{Q}$. $|f(x)|\leq b$ for all $x\in[a,b]$ and hence it is a bounded function. However the lower sum $L(f)=0$ and $U(f)=\frac{b^2-a^2}{2}$ (or $b-a$) $\neq 0$ since $a<b$, which implies that the function is not Riemann integrable.

Q2. *(15 marks)* Let $f:[a,b]\to\mathbb{R}$ be a bounded function and

$$\underline{I}=\underline{\int_a^b} f>0. \tag{1}$$

Prove that there exists an interval $I\subset[a,b]$ such that $f>0$ on I.

Proof: Given that f is a bounded function. Let $M>0$ such that $|f(x)|\leq M$ for all x. If $f(x)\leq 0,\forall x$, then it contradicts the equation (1) in the hypothesis. Fix an

$$\epsilon=\frac{\underline{I}}{2(M+b-a)}>0.$$

and set $S=\{x\in[a,b]:f(x)\geq\epsilon\}$.
By definition, we have $L(P,f)\leq\underline{\int_a^b}f$ for all partitions P. By (1), for $\underline{I}/2>0$, there exists a partition $P=\{x_0,x_1,\cdots,x_n\}$ such that $\underline{I}/2<L(P,f)=\sum_i m_i(x_i-x_{i-1})$, where $m_i=\inf_{x\in[x_{i-1},x_i]}f(x)$.

Let $A=\{k:[x_{k-1},x_k]\in S\}$ and $B=\{k:k\notin A\}$. Then

$$\begin{aligned}
&\frac{\underline{I}}{2}<L(P,f)=\sum_{i\in A}m_i(x_i-x_{i-1})+\sum_{B}m_i(x_i-x_{i-1})\\
&\quad<M\sum_{i\in A}(x_i-x_{i-1})+\epsilon\sum_{i\in B}(x_i-x_{i-1})<M\sum_{i\in A}(x_i-x_{i-1})+\epsilon(b-a)\\
\implies\quad &\frac{\underline{I}}{2(M+b-a)}<\sum_{i\in A}(x_i-x_{i-1})
\end{aligned}$$

Thus S contains finite number of intervals at which f is positive. Hence the proof.
(Reference Ex 7.35 Apostol's book)

Q3. *(15 marks)* Let $f \in \mathcal{R}[a,b]$. Prove that $|f| \in \mathcal{R}[a,b]$ and

$$\Big|\int_a^b f\Big| \leq \int_a^b |f|.$$

Since $f \in \mathcal{R}[a,b]$, for a given ϵ, there is a partition $P = \{x_0, x_1, \cdots, x_n\}$ of $[a,b]$ such that $U(P,f) - L(P,f) < \epsilon$. Then

$$\begin{aligned}
\big||f(x)| - |f(y)|\big| &\leq |f(x) - f(y)| \\
\implies sup_{x,y\in[x_{i-1},x_i]}(|f(x)| - |f(y)|)\big| &\leq sup_{x,y\in[x_{i-1},x_i]}(f(x) - f(y)) \\
\implies sup_{x\in[x_{i-1},x_i]}|f(x)| - \inf_{y\in[x_{i-1},x_i]}|f(y)|\big| &\leq sup_{x\in[x_{i-1},x_i]}f(x) - inf_{y\in[x_{i-1},x_i]}f(y) \\
\implies U(P,|f|) - L(P,|f|) &\leq U(P,f) - L(P,f) < \epsilon
\end{aligned}$$

Thus $|f| \in \mathcal{R}[a,b]$.

Let $c = \pm 1$. Then $\big|\int_a^b f| = c\int_a^b f = \int_a^b cf \leq \int_a^b |f|$.

Q4. *(10 marks)* Let $f \in \mathcal{R}[a,b]$. Prove that for

$$F(x) = \int_a^x f \quad (x \in [a,b]),$$

there is $M > 0$ such that

$$|F(x) - F(y)| \leq M|x-y| \quad (x,y \in [a,b]).$$

We use the following property of Riemann integral: $\int_a^x f + \int_x^c f = \int_a^c$. Hence for $x < y$, $\int_a^x f - \int_a^y f = \int_y^x f$. Hence $|F(x) - F(y)| = \big|\int_x^y f\big|$. Since $f \in \mathcal{R}[a,b]$, f is bounded. Hence there exists $M > 0$ such that $|f(x)| \leq M$ for all $x \in [a,b]$. Thus by the previous question, $\big|\int_x^y f\big| \leq \int_x^y |f| \leq M|x-y|$. Similarly we prove for $x > y$.

Q5. *(10 marks)* Suppose that (X,d) is a metric space and $\{S_\alpha\}_{\alpha\in\Lambda}$ is a collection of subsets of X. Prove that

$$\cup_\alpha \overline{S_\alpha} \subset \overline{\cup_\alpha S_\alpha} = \overline{\cup_\alpha \overline{S_\alpha}}.$$

If x_α is a point in S_α, then clearly it belongs to $\overline{\cup_\alpha S_\alpha}$. Suppose x_α is a limit point of S_α. Since $S_\alpha \subset \cup_\alpha S_\alpha$, we have $x_\alpha \in \overline{\cup_\alpha S_\alpha}$. Thus $\cup_\alpha \overline{S_\alpha} \subset \overline{\cup_\alpha S_\alpha}$.

Also, $S_\alpha \subset \overline{S_\alpha}$ implies $\cup_\alpha S_\alpha \subset \cup_\alpha \overline{S_\alpha}$ which implies $\overline{\cup_\alpha S_\alpha} \subset \overline{\cup_\alpha \overline{S_\alpha}}$.

Yet to prove that $\overline{\cup_\alpha \overline{S_\alpha}} \subset \overline{\cup_\alpha S_\alpha}$. Let x be a point in $\cup_\alpha \overline{S_\alpha}$. Then x is a limit point of S_α for some α. Every neighborhood of x contains a point in S_α other than x and hence contains a point in $\cup_\alpha S_\alpha$ other that x. That is, x is a limit point of $\cup_\alpha S_\alpha$.
Suppose x is a limit point of $\cup_\alpha \overline{S_\alpha}$. To prove that every neighborhood ball $B_r(x)$ of x

with radius $r > 0$ contains a point $z \neq x$ such that $z \in \cup_\alpha S_\alpha$. Since x is a limit point of $\cup_\alpha \overline{S_\alpha}$, neighborhood ball $B_{\frac{r}{3}}(x)$ contains a point $y \neq x$ such that $y \in \cup_\alpha \overline{S_\alpha}$, that is, y is a limit point of some S_α. Hence the neighborhood ball $B_{\frac{r}{3}}(y)$ contains a point $z \neq y$, $z \neq x$ such that $z \in S_\alpha$. Thus

$$d(z,x) \leq d(z,y) + d(y,x) \leq \frac{2r}{3} < r \implies z \in B_r(x).$$

Since $z \in S_\alpha \subset \cup_\alpha S_\alpha$ and r is arbitrary, hence the proof.

Q6. *(15 marks)* Prove or disprove the following:

(i) A discrete metric space is complete.-**True**

Let $\{x_n\}$ be a Cauchy sequence. That is, for every $\epsilon > 0$ there exists $N > 0$ such that $d(x_n, x_m) < \epsilon$ for all $n, m \geq N$. Since $d(\ ,\ )$ is a discrete metric, $x_n = x_m$ for all n, m. Thus every Cauchy sequence is a discrete metric space is a constant sequence. Hence they converge to the constant. Thus every Cauchy sequence converges in a discrete metric space. Hence discrete metric spaces are complete.

(ii) An infinite subset of a metric space has a limit point.-**False**

Consider the metric space $X = (0,1)$ with usual Euclidean metric $d(x,y) = |x - y|$. The sequence $\{\frac{1}{n}\}_{n>1}$ converge to 0 which does not lie in X. However, every infinite subset of a compact metric space has a limit point.

(iii) A non-empty complete metric space without isolated points is uncountable.**True**

Suppose X is a non-empty complete metric space without isolated points and is countable: $X = \{x_1, x_2, \cdots\}$. Consider $U_n = X \backslash \{x_n\}$. Since X has no isolated points, U_n is dense in X for all n. By Baire Category theorem, $\cap_n U_n$ is dense in X. However, by the construction, $\cap_n U_n$ is an empty set which gives the contradiction.

Q7. *(15 marks)* Prove that $B[0,1]$ (with uniform metric) is not separable.

Let $d(f,g) = \sup_{x\in[0,1]}\{|f(x) - g(x)|\}$ denote the uniform metric in the set $B[0,1]$ of all bounded functions on $[0,1]$. Consider the uncountable family $S = \{f_x\}_{x\in[0,1]}$ of functions in $B[0,1]$ defined as follows: $\forall x, y \in [0,1]$

$$f_x(y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{elsewhere.} \end{cases}$$

Clearly f_x are bounded functions. Also, $d(f_x, f_y) = 1$ if $x \neq y$. Hence for all x, 1/2-radius neighborhoods of f_x are pairwise disjoint.
Recall that a metric space is called separable if it contains a countable dense subset. Suppose there exists a countable dense subset $D = \{g_n\}_{n\in\mathbb{N}}$ of $B[0,1]$. Then for every x,

every neighborhood of f_x should intersect D. That is, 1/2-radius neighborhood N_x of f_x contains g_{n_x} for some $n_x \in \mathbb{N}$.

$$1 = d(f_x, f_y) \leq d(f_x, g_{n_y}) + d(g_{n_y}, f_y) < d(f_x, g_{n_y}) + 1/2 \implies d(f_x, g_{n_y}) > 1/2.$$

Thus for every x we find $g_{n_x} \in D$, that is, there exists an uncountable subset of D which is a contradiction.

Q8. *(10 marks)* Prove that if $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in (X, d), then $\{d(x_n, y_n)\}$ is a convergent sequence.

Since $\{x_n\}$ and $\{y_n\}$ are Cauchy, for every $\epsilon > 0$ there exists positive integers N_1 and N_2 such that $d(x_{n_1}, x_{m_1}) < \epsilon$ and $d(y_{n_2}, y_{m_2}) < \epsilon$ for all $n_1, m_1 \geq N_1$ and $n_2, m_2 \geq N_2$. Let $N =$max$\{N_1, N_2\}$. Then we have $d_n = d(x_n, y_n) \geq 0$ for all n and

$$\begin{aligned}
d_n &= d(x_n, y_n) \\
&\leq d(x_n, x_m) + d(x_m, y_n) \\
&\leq d(x_n, x_m) + d(x_m, y_m) + d(y_m, y_n) \\
&\leq 2\epsilon + d(x_m, y_m) \forall\ n, m \geq N
\end{aligned}$$

Thus $|d_n - d_m| \leq 2\epsilon$. Thus $\{d_n\}$ is a Cauchy sequence in $\mathbb{R}$. Since $\mathbb{R}$ is complete, $\{d_n\}$ is a convergent sequence.

Q9. *(15 marks)* Prove that a closed interval cannot be expressed as the union of a countable family of disjoint nonempty closed sets.

Suppose $\{F_n\}_{n \in \mathbb{N}}$ is a countable (two or more) family of disjoint non-empty closed sets whose union is the closed interval I. Since F_1 and F_2 are two non-empty disjoint closed sets, there exists disjoint open sets U_1 and U_2 such that $F_i \subset U_i$ for all $i = 1, 2$. Let $I_1 \subset I \backslash U_1$ be a closed non-empty interval such that $I_1 \cap F_2 \neq \emptyset$. Then $I_1 \cap U_1 = \emptyset$ and hence $I_1 \cap F_1 = \emptyset$. By connectedness of the interval I_1 there exists infinitely many F_m such that $I_1 \cap F_n \neq \emptyset$. Now $I_1 = \cup_m (F_m \cap I_1)$ is a union of countably non-empty disjoint closed sets. Repeating the argument we find $I_2 \subset I_1$. Proceeding we have a decreasing sequence of nonempty closed intervals I_n such that $I_n \cap F_n = \emptyset$.
Claim: $\cap_n I_n$ is non-empty. Let $I_n = [a_n, b_n]$. Then the set E of all a_n's is bounded above by b_1 and let x be the supremum of E. Since $a_n \leq a_{n+m} \leq b_{m+n} \leq b_n$, we have $a_m \leq x \leq b_m$ for all m. Hence $x \in I_m$ for all m. Hence the claim.
Let $x \in \cup_n I_n \subset I$. However, since $I_n \cap F_n = \emptyset$, $x \notin F_n$. But $I = \cup_n F_n$ contradicts $x \in I$.